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LETTER TO THE EDITOR

von Neumann's hypothesis concerning coherent states

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Abstract
An orthonormal basis of modified coherent states is constructed. Each member of the basis is an infinite sum of coherent states on a von Neumann lattice. A single state is assigned to each unit cell of area h (Planck constant) in the phase plane. The uncertainties of the coordinate x and the square of the momentum p^2 for these states are shown to be similar to those for the usual coherent states. Expansions in the newly established set are discussed and it is shown that any function in the kq-representation can be written as a sum of two fixed kq-functions. Approximate commuting operators for x and p^2 are defined on a lattice in phase plane according to von Neumann's prescription.

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In his celebrated book '*Mathematical Foundations of Quantum Mechanics*' [1], von Neumann has constructed a set of coherent states on a lattice in the phase plane, with a unit cell of area h, the Planck constant. What we call here von Neumann's hypothesis is his statement, without proof, that the set he built is complete, and that it can be made orthogonal. The completeness part of this hypothesis was proved to be correct [2–4], about 40 years after it was first stated in the German edition of von Neumann's book. The orthogonality part of the hypothesis has taken a fascinating turn in the last 20 years after the Balian–Low theorem was proved [5, 6]. An idea to bypass the Balian–Low theorem was raised by Wilson and co-workers who designed a numerical iteration procedure for building orthonormal functions on a von Neumann lattice [7]. In this procedure, the functions in the x-direction are obtained from one another by translations, as in the von Neumann case, but on the p-axis they are doubly peaked, and cannot be obtained by translations. As such, these functions do not violate the Balian–Low theorem and can be chosen to be well localized. An analytic construction of such functions was first carried out by Danbechies *et al*, who called these functions the Wilson orthonormal basis [8]. In a recent publication [9] a symmetry framework was developed for the Wilson orthonormal basis and a unique assignment was established of each function in the basis to a unit cell of area h on the von Neumann lattice in the phase plane.

In this letter we show how the functions of the Wilson orthonormal basis can be expressed as infinite linear combinations of coherent states on a discrete lattice in the phase plane. We denote these functions by $\varphi_{\ell m}(k, q)$ and show that they are modified coherent states. (ℓ, m)

denotes a unit cell of area h in the (p, x)-plane, and there is a single $\varphi_{\ell m}(k, q)$-function assigned to the cell (ℓ, m). In this sense the $\varphi_{\ell m}(k, q)$ fit into the framework of modified coherent states, suggested by von Neumann about 70 years ago. However, since the $\varphi_{\ell m}(k, q)$ are double peaked on the p-axis, they are used in this letter to construct approximate commuting operators for x and p^2, and not for x and p as was von Neumann's goal [1]. The $\varphi_{\ell m}$ have many interesting properties, and they can very conveniently be used in expansions of different functions. For this purpose we also define biorthogonal modified coherent states, and using them the uncertainties of the coordinate x and the momentum p are calculated. In particular, we calculate Δx and $\Delta(p^2)$ in the modified coherent states and we show that they are very much the same as in the coherent states on the lattice in the phase plane. Using the results for the expectation values of x and p^2 we construct approximate commuting operators for x and p^2. In expanding the $\varphi_{\ell m}$-functions we also show that any $C(k, q)$-function can be written as a sum of coherent states $C_0(k, q)$ and $C_0\big(k, q - \frac{a}{2}\big)$ with coefficients that are periodic functions of k and q.

In dealing with the problem of macroscopic measurement in quantum mechanics, von Neumann has suggested building a discrete set of coherent states in the following way [1]:

$$\psi_{\ell m}(x) = \exp\left(\mathrm{i}\frac{2\pi}{a}x\ell\right)\psi_0(x - ma). \tag{1}$$

Here a is an arbitrary constant, $\psi_0(x)$ is the ground state of a harmonic oscillator, and ℓ and m are integers running from $-\infty$ to $+\infty$. The $\psi_{\ell m}(x)$ are coherent states centred on a lattice in the phase plane with a unit cell of area h. In [1] von Neumann stated, without proof, the completeness of the set of states in equation (1), which is very appealing in view of the fact that their density in equation (1) corresponds exactly to one state per unit cell of area h. As was mentioned above, the completeness of the set in equation (1) was proved about 40 years later [2–4]. In [1] von Neumann also stated without proof, that by using the Gramm–Schmidt orthogonalization process one can build out of the set in equation (1) 'a resulting normalized orthogonal set $\psi'_{\mu\nu}$ without any particular difficulties'. To our knowledge, this suggestion was never carried out. In what follows we construct a complete and orthonormal set of modified coherent states by using the kq-representation, which is defined in the following way [10]:

$$C(k, q) = \left(\frac{q}{2\pi}\right)^{\frac{1}{2}} \sum_n \exp(\mathrm{i}kan)\psi(x - na)$$

where $C(k, q)$ and $\psi(x)$ are the wavefunctions in the kq- and x-representations, respectively. The coordinate x and momentum p in the kq-representation are

$$x = \mathrm{i}\frac{\partial}{\partial k} + q \qquad p = \mathrm{i}\hbar\frac{\partial}{\partial q}.$$

In [9] an orthonormal Wilson-like basis was constructed, which in the kq-representation is

$$\varphi_{\ell m}(k, q) = \frac{C_{\ell m}(k, q)}{S^{1/2}(k, q)} \tag{2}$$

where

$$\begin{aligned} C_{\ell m}(k, q) &= \exp(-\mathrm{i}kam)\cos\left[\frac{2\pi}{a}\left(q - \frac{a}{4}\right)\ell\right] C_0(k, q) \\ &\quad + \mathrm{i}\exp(-\mathrm{i}kam)\sin\left[\frac{2\pi}{a}\left(q - \frac{a}{4}\right)\ell\right] C_0\left(k, q - \frac{a}{2}\right) \end{aligned} \tag{3}$$

and

$$S(k, q) = \theta_3\left(\frac{ka}{2}\middle|\mathrm{i}\frac{a^2}{4\pi\lambda^2}\right)\theta_3\left(\frac{2\pi q}{a}\middle|\mathrm{i}\frac{4\pi\lambda^2}{a^2}\right). \tag{4}$$

In equation (3) $C_0(k,q)$ is the ground state of the harmonic oscillator in the kq-representation, and a is the same arbitrary constant as in equation (1). In equation (4) $\theta_3(z|\tau)$ is the Jacobi theta function [11]

$$\theta_3(z|\tau) = \sum_n \exp(2\mathrm{i}zn + \mathrm{i}\pi\tau n^2) \tag{5}$$

and λ is the spread of the Gaussian in the ground state of the harmonic oscillator $\psi_0(x) = \left(\frac{1}{\pi\lambda^2}\right)^{\frac{1}{4}} \exp\left(-\frac{x^2}{2\lambda^2}\right)$. There is a simple connection between the functions $C_{\ell m}(k,q)$ in equation (3) and coherent states $\langle k,q|\bar{x},\bar{p}\rangle$ ($\bar{x}$ and $\bar{p}$ are the expectation values of x and p in the coherent state):

$$\begin{aligned} C_{\ell m}(k,q) = \frac{(-1)^{\ell m}}{2} & \left\{ \mathrm{i}^{\ell} \left\langle k,q \,\middle|\, ma, \ell\frac{2\pi}{a}\hbar \right\rangle + (-\mathrm{i})^{\ell} \left\langle k,q \,\middle|\, ma, -\ell\frac{2\pi}{a}\hbar \right\rangle \right. \\ & \left. + \left\langle k,q \,\middle|\, \left(m+\frac{1}{2}\right)a, \ell\frac{2\pi}{a}\hbar \right\rangle - \left\langle k,q \,\middle|\, \left(m+\frac{1}{2}\right)a, -\ell\frac{2\pi}{a}\hbar \right\rangle \right\}. \end{aligned} \tag{6}$$

Now, using the fact that $S(k,q)$ in equation (4) is periodic in k with period $\frac{2\pi}{a}$ and in q with period $\frac{a}{2}$, we can expand $\frac{1}{S^{1/2}(k,q)}$ in a Fourier series:

$$\frac{1}{S^{1/2}(k,q)} = \sum_{st} a_{st} \exp\left(-\mathrm{i}kas + \mathrm{i}q\frac{2\pi}{a}2t\right). \tag{7}$$

We would like to point out that the function $\frac{1}{S^{1/2}(k,q)}$ as a function of k and q is continuous with continuous derivatives to any order. Therefore the Fourier series in equation (7) converges uniformly at any point in the unit cell of the variation of k and q. Using the results in equations (6) and (7), we can rewrite the function $\varphi_{\ell m}(k,q)$ in equation (2) as an infinite sum of coherent states:

$$\begin{aligned} \varphi_{\ell m}(k,q) = \frac{(-1)^{\ell m}}{2} \sum_{st} (-1)^{s\ell} a_{st} & \left\{ \mathrm{i}^{\ell} \left\langle kq \,\middle|\, (m+s)a, (\ell+2t)\frac{2\pi}{a}\hbar \right\rangle \right. \\ & + (-\mathrm{i})^{\ell} \left\langle kq \,\middle|\, (m+s)a, (-\ell+2t)\frac{2\pi}{a}\hbar \right\rangle \\ & + (-1)^{t} \left\langle kq \,\middle|\, \left(m+s+\frac{1}{2}\right)a, (\ell+2t)\frac{2\pi}{a}\hbar \right\rangle \\ & \left. - (-1)^{t} \left\langle kq \,\middle|\, \left(m+s+\frac{1}{2}\right)a, (-\ell+2t)\frac{2\pi}{a}\hbar \right\rangle \right\}. \end{aligned} \tag{8}$$

The functions $\varphi_{\ell m}(k,q)$ are the same as in equation (2), and they form therefore an orthonormal basis. What equation (8) shows is that $\varphi_{\ell m}(k,q)$ can be expressed as an infinite linear combination of coherent states on a lattice in the phase plane. From what is said above about $\frac{1}{S^{1/2}(k,q)}$ (smoothness), the expansion coefficients in equation (7) (or equation (8)) decrease faster than any powers of $|s|$ and $|t|$ (see [12], p 249). From a closer look at equations (6) and (8), the following picture emerges: in equation (6) for each fixed label (ℓ, m) of the $C_{\ell m}(k,q)$, there is a sum of four coherent states around the sites

$$\left(ma, \ell\frac{2\pi}{a}\hbar\right) \qquad \left(ma, -\ell\frac{2\pi}{a}\hbar\right) \qquad \left[\left(m+\frac{1}{2}\right)a, \ell\frac{2\pi}{a}\hbar\right] \qquad \left[\left(m+\frac{1}{2}\right)a, -\ell\frac{2\pi}{a}\hbar\right]. \tag{9}$$

The function $\varphi_{\ell m}(k,q)$ then in equation (8) is obtained from the function $C_{\ell m}(k,q)$ in equation (6) by applying to the latter the operator $\frac{1}{S^{1/2}(p/\hbar,x)}$ which in turn is obtained from

$\frac{1}{S^{1/2}(k,q)}$ by replacing k by the momentum operator p and q by the coordinate operator x. We end up with the functions $\varphi_{\ell m}(k,q)$ which are infinite sums of coherent states, as described above. Being a complete and orthonormal set on a lattice in phase plane, the functions $\varphi_{\ell m}(k,q)$ fit into the general framework of von Neumann's hypothesis on coherent states [1]. However, in detail, as seen from equation (9), the functions $\varphi_{\ell m}(k,q)$ in equation (8) (like the functions $C_{\ell m}(k,q)$ in equation (6)) are well localized on the x-axis around ma, but on the p-axis they are double peaked around $\ell\frac{2\pi}{a}\hbar$ and $-\ell\frac{2\pi}{a}\hbar$. In the von Neumann hypothesis the modified coherent states are supposed to be well localized around single sites $\left(ma, \ell\frac{2\pi}{a}\hbar\right)$.

The functions $C_{\ell m}(k,q)$ (equation (6) or (3)) and $\varphi_{\ell m}(k,q)$ (equation (8) or (2)) have a number of interesting properties. One can see that $C_{\ell m}(k,q)$ form a complete set (and not overcomplete!) because they are related to $\varphi_{\ell m}(k,q)$ by the function $S^{1/2}(k,q)$ (equation (4)) that has no zeros for real k and q. However, the $C_{\ell m}(k,q)$ are, in general, not orthogonal:

$$\int C^{\star}_{\ell m}(k,q)C_{\ell' m'}(k,q)\,\mathrm{d}k\,\mathrm{d}q$$
$$= \begin{cases} 0 & \text{if} \quad \ell'-\ell = 2s+1 \quad \text{(odd)} \\ (-1)^s \exp\left\{-\frac{a^2}{4\lambda^2}(m'-m)^2 - \frac{\pi\lambda^2}{a^2}(\ell'-\ell)^2\right\} & \text{if} \quad \ell'-\ell = 2s \quad \text{(even).} \end{cases} \tag{10}$$

$C_{\ell m}(k,q)$ are orthogonal for $\ell'-\ell \equiv$ odd. It is interesting to compare this result with the similar relation for the coherent states $\left\langle kq \middle| ma, \ell\frac{2\pi}{a}\hbar\right\rangle$ on the lattice:

$$\left\langle ma, \ell\frac{2\pi}{a}\hbar \middle| m'a, \ell'\frac{2\pi}{a}\hbar\right\rangle = (-1)^{mn'+m'n} \exp\left\{-\frac{a^2}{4\lambda^2}(m'-m)^2 - \frac{\pi^2\lambda^2}{a^2}(\ell'-\ell)^2\right\}. \tag{11}$$

This shows that for $\ell'-\ell = 2s$ (even), $C_{\ell m}(k,q)$ and $\left\langle kq \middle| ma, \ell\frac{2\pi}{a}\hbar\right\rangle$ satisfy the same orthogonality relation (up to a phase), while for $\ell'-\ell = 2s+1$ (odd), the $C_{\ell m}(k,q)$ are orthogonal, which means that the modified coherent states $C_{\ell m}(k,q)$ have better orthogonality properties than the coherent states themselves.

Since the $C_{\ell m}(k,q)$ form a complete and non-orthogonal set, one can look for a biorthogonal set, which we denote by $\tilde{C}_{\ell m}(k,q)$. From the definition of $\varphi_{\ell m}(k,q)$ via the $C_{\ell m}(k,q)$ (equation (2)) and the fact that $\varphi_{\ell m}(k,q)$ is an orthonormal basis, one sees immediately that the biorthogonal set is

$$\tilde{C}_{\ell m}(k,q) = \frac{C_{\ell m}(k,q)}{S(k,q)} \tag{12}$$

because

$$\int \tilde{C}^{\star}_{\ell m}(k,q)C_{\ell' m'}(k,q)\,\mathrm{d}k\,\mathrm{d}q = \delta_{\ell\ell'}\delta_{mm'}. \tag{13}$$

Being complete sets, either $\varphi_{\ell m}(k,q)$ (equation (8)) or $C_{\ell m}(k,q)$ (equation (6)) can be used in expansions. Despite their being non-orthogonal the functions $C_{\ell m}(k,q)$ have an advantage over $\varphi_{\ell m}(k,q)$ for the following reason. $C_{\ell m}(k,q)$ (equation (6)) are sums of four coherent states. Therefore, when expanding the biorthogonal functions $\tilde{C}_{\ell m}(k,q)$, it will be simple to find the expansion coefficients. Indeed, we have for any function $C(k,q)$

$$C(k,q) = \sum_{\ell m} c_{\ell m}\tilde{C}_{\ell m}(k,q) = \sum_{\ell m} c_{\ell m}\frac{C_{\ell m}(k,q)}{S(k,q)}. \tag{14}$$

By multiplying both sides by $C^{\star}_{\ell m}(k,q)$ and using the orthogonality relation (13), we have for the coefficients $c_{\ell m}$

$$c_{\ell m} = \int C(k,q)C^{\star}_{\ell m}(k,q)\,\mathrm{d}k\,\mathrm{d}q. \tag{15}$$

Keeping in mind that $C^{\star}_{\ell m}(k,q)$ is a sum of four coherent states, it is, in general, easy to find the coefficients $c_{\ell m}$ for a large variety of functions $C(k,q)$, for which the scalar products $\left\langle ma, \ell\frac{2\pi}{a}\hbar \middle| C(k,q)\right\rangle$ are known. Thus, it is easy to find the coefficients $c_{\ell m}$ for any coherent state $\langle k,q|\bar{x},\bar{p}\rangle$ or any harmonic oscillator $\langle k,q|n\rangle$ [12].

Expansions similar to those in equation (14) can be carried out by using the orthonormal functions $\varphi_{\ell m}(k,q)$ in equation (8):

$$C(k,q) = \sum_{\ell m} d_{\ell m}\varphi_{\ell m}(k,q) \tag{16}$$

with $d_{\ell m}$ being given by

$$d_{\ell m} = \int C(k,q)\varphi^{\star}_{\ell m}(k,q)\,\mathrm{d}k\,\mathrm{d}q. \tag{17}$$

Since $\varphi_{\ell m}(k,q)$ are infinite sums of coherent states (equation (8)) it is, in general, harder to find the exact expressions for $d_{\ell m}$. However, having in mind that the coefficients a_{st} in equation (7) fall off faster than powers of $|s|$ and $|t|$, it should be possible to find approximate $d_{\ell m}$. On the other hand, the $d_{\ell m}$ have the advantage that $|d_{\ell m}|^2$ gives the probability of finding the system in the $\varphi_{\ell m}(k,q)$-state when it is described by $C(k,q)$ (equation (16)), since the set $\varphi_{\ell m}(k,q)$ is orthonormal.

Regardless of which of the expansions we use, equation (14) or equation (16), they both lead to the following interesting result, namely, that any function $C(k,q)$ in the kq-representation can be written as a sum of $C_0(k,q)$ and $C_0\left(k,q-\frac{a}{2}\right)$ (we choose the expansion in equation (14))

$$S(k,q)C(k,q) = P(k,q)C_0(k,q) + Q(k,q)C_0\left(k,q-\frac{a}{2}\right) \tag{18}$$

where $P(k,q)$ and $Q(k,q)$ are periodic functions of k with period $\frac{2\pi}{a}$ and of q with period a, given by

$$\begin{aligned} P(k,q) &= \sum_{\ell m} c_{\ell m}\exp(-\mathrm{i}kam)\cos\left[\frac{2\pi}{a}\left(q-\frac{a}{4}\right)\ell\right] \\ Q(k,q) &= \mathrm{i}\sum_{\ell m} c_{\ell m}\exp(-\mathrm{i}kam)\sin\left[\frac{2\pi}{a}\left(q-\frac{a}{4}\right)\ell\right]. \end{aligned} \tag{19}$$

The result in equation (18) follows directly from equation (3) and the expansion (14). One can check that equation (18) splits the space of all the functions $C(k,q)$ into two orthogonal subspaces containing the functions $P(k,q)C_0(k,q)$ and $Q(k,q)C_0\left(k,q-\frac{a}{2}\right)$, respectively. This can be verified by checking that the functions in the first line of equation (3) are orthogonal for all ℓ and m to the functions in the second line of equation (3). This splitting of the space of functions into two orthogonal subspaces follows from the symmetry of $P(k,q)$ and $Q(k,q)$ under the inversion operation $\left(I\middle|\frac{a}{2}\right)$

$$\left(I\middle|\frac{a}{2}\right)P(k,q) = P(-k,q) \qquad \left(I\middle|\frac{a}{2}\right)Q(k,q) = -Q(-k,q). \tag{20}$$

One can check that when $C(k,q) = C_0(k,q)$ in equation (18), the second term on the right-hand side will vanish, and $P(k,q)$ will be equal to $S(k,q)$. A more interesting case is when

$$C(k,q) = A(k,q)C_0(k,q) \tag{21}$$

where $A(k,q)$ is periodic in k with period $\frac{2\pi}{a}$ and in q with period a. Then, neither $P(k,q)$ nor $Q(k,q)$ in equation (18) vanishes, but one has to show that the following relation holds:

$$[P(k,q) - S(k,q)A(k,q)]C_0(k,q) + Q(k,q)C_0\left(k,q-\frac{a}{2}\right) = 0 \tag{22}$$

which then turns equation (18) into an identity with the right-hand side identically equal to the left-hand side. By calculating $P(k,q)$ and $Q(k,q)$ explicitly in equation (18) for the function in equation (21) (regardless of what $A(k,q)$ is), the following relation between theta functions is obtained from equation (22) (see the definition of a theta function in [11]):

$$\left[\theta_3\left(\frac{ka}{2}\middle|\rho\right)\theta_3\left(\frac{2\pi q}{a}\middle|\rho'\right)-\theta_4\left(\frac{ka}{2}\middle|\rho\right)\theta_2\left(\frac{2\pi q}{a}\middle|\rho'\right)\right]\theta_3\left(\frac{ka}{2}-\mathrm{i}\frac{qa}{2\lambda^2}\middle|2\rho\right)$$
$$=\left[\theta_2\left(\frac{ka}{2}\middle|\rho\right)\theta_4\left(\frac{2\pi q}{a}\middle|\rho'\right)-\mathrm{i}\theta_1\left(\frac{ka}{2}\middle|\rho\right)\theta_1\left(\frac{2\pi q}{a}\middle|\rho'\right)\right]\theta_2\left(\frac{ka}{2}-\mathrm{i}\frac{qa}{2\lambda^2}\middle|2\rho\right) \tag{23}$$

where $\rho=\mathrm{i}\frac{a^2}{4\pi\lambda^2}$ and $\rho'=\mathrm{i}\frac{4\pi\lambda^2}{a^2}$. We do not find in the literature the relation among theta functions in equation (23) but we do find some special cases of it, for example at $k=q=0$ [13]. An example of a function in equation (21) is discussed in [14].

Having shown how the functions $C_{\ell m}(k,q)$ (equation (3) or (6)) and $\varphi_{\ell m}(k,q)$ (equation (2) or (8)) can be expanded in coherent states, we will now turn to the uncertainties of the coordinate x and momentum p. This will show us in what sense it is justified to call $C_{\ell m}(k,q)$ or $\varphi_{\ell m}(k,q)$ the modified coherent states. We have already pointed out that $C_{\ell m}(k,q)$ is a sum of four coherent states (equations (6) and (9)). From the point of view of x the first two states in equation (6) are located around ma, and the other two states around $\left(m+\frac{1}{2}\right)a$. This shows that the uncertainty Δx in the state $C_{\ell m}(k,q)$ should not differ much from the corresponding uncertainty Δx for a coherent state. The situation is different for p because the states in equation (9) contain coherent states with both $\ell\frac{2\pi}{a}\hbar$ and $-\ell\frac{2\pi}{a}\hbar$. One should therefore expect that Δp for $C_{\ell m}$ will be different from that obtained for a coherent state and we will not calculate Δp here. However, equation (9) shows that one should expect that the uncertainty of the square of the momentum $\Delta(p^2)$ in the state $C_{\ell m}(k,q)$ should not differ much from the corresponding uncertainty of p^2 in a coherent state. Calculations of Δx and $\Delta(p^2)$ confirm these expectations. It is straightforward to show that in the $C_{\ell m}(k,q)$-state one has for $\frac{a^2}{2\pi\lambda^2}=1$ and for $\ell\neq 0$ (for $\ell=0$, $C_{\ell m}(k,q)$ turns into a coherent state)

$$\langle x\rangle = a\left(m+\frac{1}{4}\right) \qquad \langle x^2\rangle = a^2\left(m^2+\frac{m}{2}+\frac{1}{4\pi}+\frac{1}{8}\right) \tag{24}$$

$$\langle p^2\rangle = \frac{h^2}{a^2}\left(\ell^2+\frac{1}{4\pi}\right) \qquad \langle p^4\rangle = \frac{h^4}{a^4}\left(\ell^4+\frac{3}{2\pi}\ell^2+\frac{3}{16\pi^2}\right). \tag{25}$$

Respectively, for the uncertainties Δx and $\Delta(p^2)$ one obtains

$$\Delta x = \frac{a}{2\sqrt{\pi}}\left(1+\frac{\pi}{4}\right)^{1/2} \qquad \Delta(p^2) = \frac{h^2}{2\sqrt{\pi}a^2}(1+8\pi\ell^2)^{1/2} \approx \frac{h^2}{\sqrt{\pi}a^2}|\ell|. \tag{26}$$

The results in equation (24)–(26) are given up to terms of order $\exp(-\pi)$.[1] It is interesting to compare these results with the uncertainties in coherent states (equation (1)). For the latter $\Delta x=\frac{a}{2\sqrt{\pi}}$ while $\Delta(p^2)$ coincides with the expression in equation (26). This shows that from the point of view of the uncertainties of x and p^2 the $C_{\ell m}(k,q)$ are close to the coherent states $\left|ma,\ell\frac{2\pi}{a}\hbar\right\rangle$. A similar statement can be made with respect to $\varphi_{\ell m}(k,q)$ (equation (8)), in view of the fact that the coefficients a_{st} fall off faster than powers of $|s|$ and $|t|$. With this in mind we call $C_{\ell m}(k,q)$ and $\varphi_{\ell m}(k,q)$ the modified coherent states. We just point out that the uncertainties Δp in the modified coherent states grow proportionally to ℓ (see equation (9)).

[1] We would like to point out that in [9] there is an error: after equation (27) it should say 'where we neglect terms of the order of $\exp(-2\pi)$'.

Let us now come back to the expansion in equation (16), and denote $\varphi_{\ell m}(k, q)$ by the ket $|\ell, m\rangle$. The orthonormality condition on $\varphi_{\ell m}(k, q)$ will then be

$$\langle \ell, m|\ell', m'\rangle = \delta_{\ell\ell'}\delta_{mm'}. \tag{27}$$

In addition, the projection operators $P(\ell m) = |\ell m\rangle\langle \ell m|$ are what is called exhaustive (complete) and exclusive [15]:

$$\sum_{\ell,m} P(\ell, m) = 1 \qquad P(\ell, m)P(\ell', m') = P(\ell, m)\delta_{\ell\ell'}\delta_{mm'}. \tag{28}$$

Equation (28) can also be looked at as a decomposition of the unit operator in the projection operators $|\ell, m\rangle\langle \ell, m|$ on the states $\varphi_{\ell m}(k, q)$. Using the decomposition in equation (28), any state $|\,\rangle$ can be written as

$$|\,\rangle = \sum_{\ell,m} \langle \ell, m|\rangle |\ell, m\rangle \tag{29}$$

and any operator O as

$$O = \sum_{\ell,m\ell'm'} |\ell, m\rangle\langle \ell, m|O|\ell', m'\rangle\langle \ell', m'|. \tag{30}$$

The result in equation (30) can be used for approximating operators on the von Neumann lattice in the phase plane. Since the states $|\ell, m\rangle$ are well localized from the point of view of x and p^2, these latter operators can be written in the lowest approximation in the following way (we use the expectation values in equations (24) and (25)):

$$X = a\sum_m m|o, m\rangle\langle o, m| + a\sum_{\ell\neq o,m}\left(m + \frac{1}{4}\right)|\ell, m\rangle\langle \ell, m| \tag{31}$$

$$P^2 = \frac{h^2}{a^2}\sum_{\ell,m}\left(\ell^2 + \frac{1}{4\pi}\right)|\ell, m\rangle\langle \ell, m|. \tag{32}$$

The first term in equation (31) follows from the fact that the expectation value $\langle x\rangle$ in the state $|o, m\rangle$ is ma. It is easy to see that X and P^2 commute, and we can call them the classical approximation for the operators x and p^2. The formula in equation (30) enables one to go to higher approximations for x and p^2.

Finally, we turn to the physical interpretation of the coefficients $d_{\ell m}$ in the expansion of equation (16). It was already pointed out that $|d_{\ell m}|^2$ gives the probability of finding the system in the $\varphi_{\ell m}(k, q)$-state. In view of the results in equations (31) and (32), we can now also say that

$$|d_{\ell,m}|^2 + |d_{-\ell,m}|^2 \tag{33}$$

is the probability of measuring the eigenvalues of the commuting operators X and P^2.

In conclusion, we have defined an orthonormal basis of modified coherent states. Each member in this set is an infinite linear combination of coherent states, and it is assigned to a single cell of area h in the phase plane. The uncertainties of x and p^2 in the newly constructed states were shown to be similar to those in regular coherent states. This fact was used to construct approximate commuting operators for x and p^2 (or kinetic energy) on the lattice in the phase plane.

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